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ON EQUATIONS OF MOTION OF A NONLINEAR HYDROELASTIC STRUCTURE

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Formal derivation of equations of a nonlinear hydroelastic structure, which is a volume of an ideal incompressible fluid covered by a shell, is proposed. The study is based on two assumptions. The first assumption implies that the energy stored in the shell is completely determined by the mean curvature and by the elementary area. In a three-dimensional case, the energy stored in the shell is chosen in the form of the Willmore functional. In a two-dimensional case, a more generic form of the functional can be considered. The second assumption implies that the equations of motion have a Hamiltonian structure and can be obtained from the Lagrangian variational principle. In a two-dimensional case, a condition for the hydroelastic structure is derived, which relates the external pressure and the curvature of the elastic shell.

Key words: *free boundary, variational principle, ideal fluid, hydroelasticity, constraint forces, Antman equation, Bernoulli law.*

1. Equations of Motion in the Lagrangian Coordinates. A long list of publications on the theory of nonlinear hydroelasticity can be found in [1].

The following notation is needed to formulate the model of a hydroelastic structure, which was first proposed in [2] to describe waves on the surface of a liquid covered by an ice layer.

Let an ideal incompressible fluid at the time t occupy a domain Ω_t in the Euclidean space of points $\boldsymbol{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$. In turn, the shell thickness is assumed to be small, and its mid-surface coincides with the boundary of the flow domain as geometric positions of points.

We consider the Lagrangian variables $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$ determining the positions of material particles. Actually, the coordinate $\boldsymbol{\xi}$ is a label of a material particle chosen more or less arbitrarily.

We assume that the points $\boldsymbol{\xi}$ occupy a certain domain $\Omega \subset \mathbb{R}^3$ with a smooth boundary Σ. Then, the positions of the fluid points are characterized by the vector field of displacements $\boldsymbol{x}(t, \boldsymbol{\xi})$ ($\boldsymbol{\xi} \in \Omega$), and the positions of the shell particles are characterized by the field of displacements $\boldsymbol{y}(t, \boldsymbol{\xi})$ ($\boldsymbol{\xi} \in \Sigma$).

In the initial-boundary problems, it is convenient to consider $\boldsymbol{\xi}$ as the positions of material points at the time $t = 0$. In this case, we have $\Omega_0 = \Omega$ and $\partial\Omega_0 = \Sigma$. Thus, the boundary of the flow domain and the shell admit two presentations:

$$\Sigma_t^{\boldsymbol{x}}\colon \quad \boldsymbol{x} = \boldsymbol{x}(t, \boldsymbol{\xi}), \qquad \Sigma_t^{\boldsymbol{y}}\colon \quad \boldsymbol{y} = \boldsymbol{y}(t, \boldsymbol{\xi}) \qquad \text{for} \quad \boldsymbol{\xi} \in \Sigma.$$

During the joint motion in the general case, the fluid may separate from the shell; hence, the surfaces $\Sigma_t^{\boldsymbol{x}}$ and $\Sigma_t^{\boldsymbol{y}}$ may fail to coincide. This effect is called the partial filling of the cavity by the fluid. In the present paper, the effect is ignored, and further considerations are limited to the case with $\Sigma_t^{\boldsymbol{x}} = \Sigma_t^{\boldsymbol{y}}$. The shell, however, may slip with respect to the ideal incompressible fluid, which means that $\boldsymbol{x}(t, \boldsymbol{\xi}) \neq \boldsymbol{y}(t, \boldsymbol{\xi})$ for $\boldsymbol{\xi} \in \Sigma$.

Let us recall the basic facts from the theory of surfaces. If the surface Σ locally admits parametrization $\boldsymbol{\xi} = \boldsymbol{\xi}(\vec{q}) = \boldsymbol{\xi}(q_1, q_2)$, then the normal vector $\boldsymbol{n}$ and the elementary surface area Σ have the following form in the coordinates (q_1, q_2):

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$$\boldsymbol{n}(\boldsymbol{\xi}(\vec{q})) = \frac{1}{\sqrt{g_0}}\,\partial_{q_1}\boldsymbol{\xi}(\vec{q}) \times \partial_{q_2}\boldsymbol{\xi}(\vec{q}), \quad d\Sigma = \sqrt{g_0}\,d\vec{q}, \quad \sqrt{g_0} = \Big|\partial_{q_1}\boldsymbol{\xi}(\vec{q}) \times \partial_{q_2}\boldsymbol{\xi}(\vec{q})\Big|.$$

If, at each time t, the moving surface $\Sigma_t^{\boldsymbol{y}}$ admits parametrization

$$\boldsymbol{Y}(t,\vec{q}) = \boldsymbol{y}(t,\boldsymbol{\xi}(\vec{q})),$$

then the vector of the outward normal to $\Sigma_t^{\boldsymbol{y}}$ and the elementary area are described by the formulas

$$\begin{gathered}\boldsymbol{\nu}(t,\boldsymbol{y}(t,\boldsymbol{\xi}(\vec{q}))) = \frac{1}{\sqrt{g_t^{\boldsymbol{y}}}}\,\partial_{q_1}\boldsymbol{Y}(t,\vec{q}) \times \partial_{q_2}\boldsymbol{Y}(t,\vec{q}), \qquad d\Sigma_t^{\boldsymbol{y}} = \sqrt{g_t^{\boldsymbol{y}}}\,d\vec{q}, \\ \sqrt{g_t^{\boldsymbol{y}}} = \Big|\partial_{q_1}\boldsymbol{Y}(t,\vec{q}) \times \partial_{q_2}\boldsymbol{Y}(t,\vec{q})\Big|.\end{gathered} \tag{1}$$

The components of the metric tensor g_{ij} ($1 \leqslant i$ and $j \leqslant 2$) and the components of the second quadratic form L_{ij} ($1 \leqslant i$ and $j \leqslant 2$) are given by the equalities

$$g_{ij} = (\boldsymbol{Y}_i, \boldsymbol{Y}_j), \quad L_{ij} = (\boldsymbol{\nu}, \partial_{q_i}\boldsymbol{Y}_j), \quad \boldsymbol{Y}_i = \partial_{q_i}\boldsymbol{Y}, \quad \boldsymbol{\nu} = \frac{\boldsymbol{Y}_1 \times \boldsymbol{Y}_2}{|\boldsymbol{Y}_1 \times \boldsymbol{Y}_2|}.$$

The doubled mean curvature H is calculated by the formula

$$H = g^{ij}L_{ij}, \tag{2}$$

where $g^{ij} = (g_{ij})^{-1}$.

Similar formulas describe the surface $\Sigma_t^{\boldsymbol{x}}$, which coincides with $\Sigma_t^{\boldsymbol{y}}$ as the geometric set of points.

The motion of a nonlinear hydroelastic structure is characterized by the velocity fields

$$\boldsymbol{v}(t,\boldsymbol{\xi}) = \partial_t\boldsymbol{x}(t,\boldsymbol{\xi}) \quad \text{for} \quad \boldsymbol{\xi} \in \Omega, \qquad \boldsymbol{u}(t,\boldsymbol{\xi}) = \partial_t\boldsymbol{y}(t,\boldsymbol{\xi}) \quad \text{for} \quad \boldsymbol{\xi} \in \partial\Omega,$$

where $\boldsymbol{v}$ is the velocity of the fluid particle and $\boldsymbol{u}$ is the velocity of the shell particle in the Lagrangian coordinates ξ_i ($i = 1, 2, 3$). In addition, the motion is characterized by the density distributions in the corresponding components.

Without losing generality, we assume that the fluid density equals unity. The shell bounding the fluid is compressible; therefore, it is necessary to use the formula for the density distribution in the shell. Let the density distribution in the shell at the initial time in the Lagrangian coordinates be defined by the function $\varrho_0(\boldsymbol{\xi})$. This means that the mass of an arbitrary part of the shell $A \subset \Sigma$ is determined by the equality

$$\int_A \varrho_0(\boldsymbol{\xi})\,d\Sigma.$$

The law of conservation of mass implies the equality

$$\int_A \varrho_0(\boldsymbol{\xi})\,d\Sigma = \int_{\boldsymbol{y}(t,A)} \varrho(t,\boldsymbol{\xi})\,d\Sigma_t^{\boldsymbol{y}},$$

which, in turn, yield the presentation

$$\varrho(t,\boldsymbol{\xi}) = \varrho_0(\boldsymbol{\xi})\Big(\frac{d\Sigma_t^{\boldsymbol{y}}}{d\Sigma}\Big)^{-1} = \varrho_0(\boldsymbol{\xi})\sqrt{\frac{g_0(\vec{q})}{g_t^{\boldsymbol{y}}(\vec{q})}}. \tag{3}$$

For simplicity, we assume that the mass is uniformly distributed in the shell at the initial time, i.e., $\varrho(0,\boldsymbol{\xi}) = \varrho_0(\boldsymbol{\xi}) = 1$. Under these assumptions, the kinetic energy of the fluid K_f and the kinetic energy of the elastic shell K_e have the form

$$K_f = \frac{1}{2}\int_\Omega |\partial_t\boldsymbol{x}(t,\boldsymbol{\xi})|^2\,d\boldsymbol{\xi},$$

$$K_e = \frac{1}{2}\int_{\Sigma_t^{\boldsymbol{y}}} \varrho(t,\boldsymbol{\xi})|\partial_t\boldsymbol{y}(t,\boldsymbol{\xi})|^2\,d\Sigma_t^{\boldsymbol{y}} = \frac{1}{2}\int_\Sigma |\partial_t\boldsymbol{y}(t,\boldsymbol{\xi})|^2\,d\Sigma.$$

Let us put forward the following hypotheses.

1. Equations that describe the nonlinear hydroelastic structure form a dynamic system with a configuration space $(\boldsymbol{x}(t,\cdot),\boldsymbol{y}(t,\cdot)) \in \Lambda \subset L^2(\Omega)^3 \times L^2(\Sigma)^3$ for $t \in (0,T)$.

2. The Lagrangian for the fluid has the form

$$L_f = \frac{1}{2}\int_\Omega |\partial_t \boldsymbol{x}(t,\boldsymbol{\xi})|^2\, d\boldsymbol{\xi}.$$

3. The Lagrangian for the shell has the form

$$L_e = \frac{1}{2}\int_\Sigma |\partial_t \boldsymbol{y}(t,\boldsymbol{\xi})|^2\, d\Sigma - \tilde{W}(\Sigma_t^{\boldsymbol{y}}),$$

where $\tilde{W}(\Sigma_t^{\boldsymbol{y}})$ is the stored (potential) elastic energy of the shell, which is defined in the form of a surface integral

$$\tilde{W}(\Sigma_t^{\boldsymbol{y}}) = \frac{1}{2}\int_{\Sigma_t^{\boldsymbol{y}}} W(\sqrt{g_t^{\boldsymbol{y}}}, |\boldsymbol{H}|)\, d\Sigma_t^{\boldsymbol{y}}; \tag{4}$$

the vector of the mean curvature $\boldsymbol{H} = H\boldsymbol{\nu}$ and the unit vector of the outward normal $\boldsymbol{\nu}$ are described by Eqs. (1) and (2); $d\Sigma_t^{\boldsymbol{y}}$ is the elementary surface area. This model of the shell is used in the nonlinear theory of elastic shells (see [3]). It should be noted that the functional $\tilde{W}(\Sigma_t^{\boldsymbol{y}})$ depends on the choice of the Lagrangian coordinates and changes its form if the independent variables are replaced. Thus, presentation (4) depends on the choice of the coordinate $\boldsymbol{\xi}$ (this issue requires careful consideration in each particular case). In the linear theory of elasticity, this problem does not arise, because the Lagrangian coordinates in this theory are chosen uniquely as the positions of particles in a certain unloaded state. It is of interest to consider the case where the functional of the stored energy is a geometric invariant and does not depend on parametrization chosen. In the class of functionals of the form (4), there exists only one geometrically invariant representative with a nontrivial dependence on the external curvature, namely, the so-called Willmore functional (see [4]):

$$\tilde{W}(\Sigma_t^{\boldsymbol{y}}) = \frac{1}{2}\int_{\Sigma_t^{\boldsymbol{y}}} |\boldsymbol{H}|^2\, d\Sigma_t^{\boldsymbol{y}}.$$

The role of the Willmore functional in the elasticity theory was noted, e.g., in [3].

4. To derive the equations of motion, we need to describe all constraints imposed on the mechanical system of motion. It is further assumed that there are two natural constrains. The first constraint is the principle of fluid incompressibility, which is written as the equation

$$\det D_\xi \boldsymbol{x}(t,\boldsymbol{\xi}) \equiv 1 \qquad \text{for} \quad \boldsymbol{\xi} \in \Omega, \tag{5}$$

where $D_\xi \boldsymbol{x}$ is the Jacobi matrix of the mapping $\boldsymbol{\xi} \mapsto \boldsymbol{x}(t,\boldsymbol{\xi})$. The second constraint reflects the coincidence of the fluid surface and elastic shell as subsets of the Euclidean space in the course of their motion:

$$\Sigma_t^{\boldsymbol{x}} = \Sigma_t^{\boldsymbol{y}}. \tag{6}$$

2. Configuration Manifold Θ. Let us consider the hydroelastic structure as a dynamic system in a linear space Λ consisting of infinitely differentiable vector fields $(\boldsymbol{x}(\cdot),\boldsymbol{y}(\cdot))$, where $\boldsymbol{x} : \Omega \to \mathbb{R}^3$ and $\boldsymbol{y} : \Sigma \to \mathbb{R}^3$. We assume that Λ has a Hilbertian structure $L^2(\Omega)^3 \times L^2(\Sigma)^3$. Under these assumptions, constraints (5) and (6) determine an infinite-dimensional configuration manifold $\Theta \subset \Lambda$. Having an induced metrics $L^2(\Omega)^3 \times L^2(\Sigma)^3$, the space Λ is not complete, and all further considerations have a formal character. The following lemma offers a description of the tangential space to Θ at the point $(\boldsymbol{x},\boldsymbol{y}) \in \Theta$.

Lemma 1. *The tangential space to the manifold Θ at the point $(\boldsymbol{x},\boldsymbol{y})$ consists of all vector fields $[\delta\boldsymbol{x}(\cdot),\delta\boldsymbol{y}(\cdot)]$, $\delta\boldsymbol{x}$: $\Omega \to \mathbb{R}^3$ and $\delta\boldsymbol{y}$: $\Sigma \to \mathbb{R}^3$, satisfying the equalities*

$$\operatorname{div}(M^{-1}\delta\boldsymbol{x}) = 0 \qquad \textit{for} \quad \boldsymbol{\xi} \in \Omega; \tag{7}$$

$$\boldsymbol{\nu}(\boldsymbol{x}(\Phi(\boldsymbol{\xi}))) \cdot \delta\boldsymbol{x}(\Phi(\boldsymbol{\xi})) = \boldsymbol{\nu}(\boldsymbol{y}(\boldsymbol{\xi})) \cdot \delta\boldsymbol{y}(\boldsymbol{\xi}) \qquad \textit{for} \quad \boldsymbol{\xi} \in \Sigma, \tag{8}$$

where $M = D_\xi \boldsymbol{x}(\boldsymbol{\xi})$ is the Jacobi matrix of the mapping $\boldsymbol{\xi} \mapsto \boldsymbol{x}(\boldsymbol{\xi})$, and the diffeomorphism $\Phi(\cdot)$: $\Sigma \mapsto \Sigma$ is defined by the equality

$$\boldsymbol{x}(\Phi(\boldsymbol{\xi})) = \boldsymbol{y}(\boldsymbol{\xi}) \qquad \textit{for} \quad \boldsymbol{\xi} \in \Sigma. \tag{9}$$

Note that the vector $(\partial_t \boldsymbol{x}, \partial_t \boldsymbol{y})$ for the trajectory of the dynamic system $[\boldsymbol{x}(t), \boldsymbol{y}(t)]$ belongs to the tangential space $\mathrm{Tan}_{(\boldsymbol{x},\boldsymbol{y})}\,\Theta$; hence, conditions (8) and (9) yield the relation

$$\boldsymbol{\nu}(t, \boldsymbol{x}(t, \Phi(t, \boldsymbol{\xi}))) \cdot \partial_t \boldsymbol{x}(t, \Phi(t, \boldsymbol{\xi})) = \boldsymbol{\nu}(t, \boldsymbol{y}(t, \boldsymbol{\xi})) \cdot \partial_t \boldsymbol{y}(t, \boldsymbol{\xi}),$$

where $\boldsymbol{x}(t, \Phi(t, \boldsymbol{\xi})) = \boldsymbol{y}(t, \boldsymbol{\xi})$ for $\boldsymbol{\xi} \in \Sigma$.

Proof of Lemma 1. Let $(\boldsymbol{x}, \boldsymbol{y}) \in \Theta$. We assume that $M(\boldsymbol{\xi}) = D_\xi \boldsymbol{x}(\boldsymbol{\xi})$ and $|M| = \det M = 1$. Then, the variation of the constraint equation (5) with respect to $\boldsymbol{x}$ has the form

$$\delta |M| = \mathrm{div}_\xi (M^{-1} \delta \boldsymbol{x}) = 0 \qquad \text{for} \quad \xi \in \Omega,$$

whence there follows Eq. (7).

We introduce auxiliary functions $\boldsymbol{x}_\mu = \boldsymbol{x}_\mu(\boldsymbol{\xi})$ and $\boldsymbol{y}_\mu = \boldsymbol{y}_\mu(\boldsymbol{\xi})$ depending on the parameter $\mu \in (-1, 1)$ and generating the surfaces $\Sigma^{\boldsymbol{x}}_\mu = \boldsymbol{x}_\mu(\Sigma)$ and $\Sigma^{\boldsymbol{y}}_\mu = \boldsymbol{y}_\mu(\Sigma)$. The equality $\Sigma^{\boldsymbol{x}} = \Sigma^{\boldsymbol{y}}$ allows the following conditions to be imposed on the functions $\boldsymbol{x}_\mu$ and $\boldsymbol{y}_\mu$:

$$\boldsymbol{x}_0 = \boldsymbol{x}, \qquad \boldsymbol{y}_0 = \boldsymbol{y}, \qquad \Sigma^{\boldsymbol{x}}_\mu = \Sigma^{\boldsymbol{y}}_\mu.$$

Hence, there exists a μ-parametric family of mappings Φ_μ: $\Sigma \to \Sigma$ such that

$$\boldsymbol{x}_\mu(\Phi_\mu(\boldsymbol{\xi})) = \boldsymbol{y}_\mu(\boldsymbol{\xi}) \qquad \text{for} \quad \boldsymbol{\xi} \in \Sigma. \tag{10}$$

In terms of the theory of Riemann manifolds, the functions $\boldsymbol{x} = \boldsymbol{x}(\boldsymbol{\xi})$ and $\boldsymbol{y} = \boldsymbol{y}(\boldsymbol{\xi})$ are called isometric immersions (see, e.g., [5]); $\boldsymbol{x}_\mu$ and $\boldsymbol{y}_\mu$ are the first-order infinitely small bendings of the immersions $\boldsymbol{x}$ and $\boldsymbol{y}$, respectively:

$$\boldsymbol{x}_\mu = \boldsymbol{x} + \mu\,\delta \boldsymbol{x} + o(\mu), \quad \boldsymbol{y}_\mu = \boldsymbol{y} + \mu\,\delta \boldsymbol{y} + o(\mu), \quad o(\mu)/\mu \to 0 \quad \text{for} \quad \mu \to 0$$

($\delta \boldsymbol{x}$ and $\delta \boldsymbol{y}$ are the first-order bending fields). By fixing an arbitrary value of $\boldsymbol{\xi} \in \Sigma$ and differentiating Eq. (10) with respect to μ at $\mu = 0$, we obtain the relation between the bending fields

$$D_\xi \boldsymbol{x}(\Phi_0(\boldsymbol{\xi}))\, \frac{d\Phi_\mu}{d\mu}(\boldsymbol{\xi})\Big|_{\mu=0} + \delta \boldsymbol{x}(\Phi_0(\boldsymbol{\xi})) = \delta \boldsymbol{y}(\boldsymbol{\xi}). \tag{11}$$

As the set $\{\Phi_\mu(\boldsymbol{\xi})\}_{\mu \in (-1,1)}$ is the curve on Σ, then $(d\Phi_\mu/d\mu)(\boldsymbol{\xi})\Big|_{\mu=0}$ is the tangential vector to Σ at the point $\Phi_0(\boldsymbol{\xi})$. Hence, $D_\xi \boldsymbol{x}(\Phi_0(\boldsymbol{\xi}))(d\Phi_\mu/d\mu)(\boldsymbol{\xi})\Big|_{\mu=0}$ is the tangential vector to $\Sigma^{\boldsymbol{x}}_0$ at the point $\boldsymbol{x}(\Phi_0(\boldsymbol{\xi}))$. Multiplying Eq. (11) in a scalar manner by the normal vector to $\Sigma^{\boldsymbol{x}}_0$ at the point $\boldsymbol{x}(\Phi_0(\boldsymbol{\xi})) = \boldsymbol{y}(\boldsymbol{\xi})$, we obtain Eq. (8). In what follows, we denote the diffeomorphism Φ_0: $\Sigma \to \Sigma$ by Φ.

As a consequence of Lemma 1, we obtain the following statement for the structure of the space $(\mathrm{Tan}_{(\boldsymbol{x},\boldsymbol{y})}\,\Theta)^\perp$ orthogonal to the manifold Θ at the point $(\boldsymbol{x}, \boldsymbol{y})$. Note, by virtue of the choice of the configuration space Θ, the space orthogonal to $\mathrm{Tan}_{(\boldsymbol{x},\boldsymbol{y})}\,\Theta$ at the point $(\boldsymbol{x}, \boldsymbol{y}) \in \Theta$ consists of all vector fields $[\boldsymbol{N}(\cdot), \boldsymbol{L}(\cdot)]$, $\boldsymbol{N}$: $\Omega \to \mathbb{R}^3$ and $\boldsymbol{L}$: $\Sigma \to \mathbb{R}^3$, satisfying the relation

$$\int\limits_\Omega \boldsymbol{N}(\boldsymbol{\xi}) \cdot \delta \boldsymbol{x}(\boldsymbol{\xi})\, d\boldsymbol{\xi} + \int\limits_\Sigma \boldsymbol{L}(\boldsymbol{\xi}) \cdot \delta \boldsymbol{y}(\boldsymbol{\xi})\, d\Sigma = 0 \qquad \forall\, (\delta \boldsymbol{x}, \delta \boldsymbol{y}) \in \mathrm{Tan}_{(\boldsymbol{x},\boldsymbol{y})}\,\Theta. \tag{12}$$

The vector fields $\boldsymbol{N}$ and $\boldsymbol{L}$ are called the constraint forces.

Lemma 2. *For each point* $(\boldsymbol{x}, \boldsymbol{y}) \in \Theta$*, the space* $(\mathrm{Tan}_{(\boldsymbol{x},\boldsymbol{y})}\,\Theta)^\perp$ *consists of the vector fields* $(\boldsymbol{N}, \boldsymbol{L})$ *such that*

$$\boldsymbol{N}(\boldsymbol{\xi}) = (M^{-1})^* \nabla_\xi p(\boldsymbol{\xi}) \qquad \textit{for} \quad \boldsymbol{\xi} \in \Omega,$$

$$\boldsymbol{L}(\boldsymbol{\xi}) = -\frac{1}{\varrho(\boldsymbol{\xi})}\,(p(\Phi(\boldsymbol{\xi})) + C)\boldsymbol{\nu}(\boldsymbol{y}(\boldsymbol{\xi})) \qquad \textit{for} \quad \boldsymbol{\xi} \in \Sigma.$$

Proof. Let $\boldsymbol{h} \in C^\infty_0(\Omega)$ and $\mathrm{div}_\xi\, \boldsymbol{h} = 0$. We choose $(\delta \boldsymbol{x}, \delta \boldsymbol{y}) \in \mathrm{Tan}_{(\boldsymbol{x},\boldsymbol{y})}\,\Theta$ in the following form: $\delta \boldsymbol{x}(\boldsymbol{\xi}) = M(\boldsymbol{\xi})\boldsymbol{h}(\boldsymbol{\xi})$ and $\delta \boldsymbol{y}(\boldsymbol{\xi}) = 0$. Hence, Eq. (12) takes the form

$$\int\limits_\Omega \boldsymbol{N}(\boldsymbol{\xi}) \cdot (M(\boldsymbol{\xi})\boldsymbol{h}(\boldsymbol{\xi}))\, d\boldsymbol{\xi} = 0.$$

As $\boldsymbol{h}$ is arbitrary, there exists a function $p(\boldsymbol{\xi})$ such that

$$\boldsymbol{N}(\boldsymbol{\xi}) = (M^*)^{-1}\nabla_\xi p(\boldsymbol{\xi}). \tag{13}$$

We choose an arbitrary vector $\boldsymbol{l}(\boldsymbol{\xi})$ orthogonal to the normal vector $\boldsymbol{\nu}(\boldsymbol{y}(\boldsymbol{\xi}))$. In Eq. (12), we assume that $\delta\boldsymbol{x}(\boldsymbol{\xi}) = 0$ and $\delta\boldsymbol{y}(\boldsymbol{\xi}) = \boldsymbol{l}(\boldsymbol{\xi})$. All tangential vectors $\boldsymbol{l}$ obey the equality

$$\int\limits_{\Sigma} \boldsymbol{L}(\boldsymbol{\xi}) \cdot \boldsymbol{l}(\boldsymbol{\xi})\, d\Sigma = 0,$$

which yields the presentation of the constraint force

$$\boldsymbol{L}(\boldsymbol{\xi}) = \lambda(\boldsymbol{\xi})\, \boldsymbol{\nu}(\boldsymbol{y}(\boldsymbol{\xi})). \tag{14}$$

We have to determine the form of the scalar function $\lambda = \lambda(\boldsymbol{\xi})$. Let $\boldsymbol{k}$ be an arbitrary solenoidal vector field $C^\infty(\Omega)$ and $\delta\boldsymbol{x} = M(\boldsymbol{\xi})\boldsymbol{k}(\boldsymbol{\xi})$ for $\boldsymbol{\xi} \in \Omega$ and $\delta\boldsymbol{y}(\boldsymbol{\xi}) = ((M \circ \Phi)(\boldsymbol{\xi})\boldsymbol{k} \circ \Phi(\boldsymbol{\xi}) \cdot \boldsymbol{\nu}(\boldsymbol{x} \circ \Phi(\boldsymbol{\xi})))\, \boldsymbol{\nu}(\boldsymbol{x} \circ \Phi(\boldsymbol{\xi}))$ for $\boldsymbol{\xi} \in \Sigma$.

Taking into account the equality $\boldsymbol{\nu}(\boldsymbol{x} \circ \Phi(\boldsymbol{\xi})) = \boldsymbol{\nu}(\boldsymbol{y}(\boldsymbol{\xi}))$ for $\boldsymbol{\xi} \in \Sigma$ and substituting presentations (13) and (14) of the constraint forces $\boldsymbol{N}$ and $\boldsymbol{L}$ to Eq. (12), we obtain the equality

$$\int\limits_{\Omega} (M\boldsymbol{k}) \cdot ((M^*)^{-1}\nabla_\xi p)\, d\boldsymbol{\xi} + \int\limits_{\Sigma} \Big(\lambda(\boldsymbol{\xi})((M \circ \Phi)\boldsymbol{k} \circ \Phi) \cdot \boldsymbol{\nu}(\boldsymbol{x} \circ \Phi)\Big)\boldsymbol{\nu}(\boldsymbol{x} \circ \Phi) \cdot \boldsymbol{\nu}(\boldsymbol{x} \circ \Phi)\, d\Sigma = 0. \tag{15}$$

As $\operatorname{div}_\xi \boldsymbol{k} = 0$, we have

$$\int\limits_{\Omega} (M\boldsymbol{k}) \cdot ((M^*)^{-1}\nabla_\xi p)\, d\boldsymbol{\xi} = \int\limits_{\Sigma} p(\boldsymbol{\xi})\boldsymbol{k} \cdot \boldsymbol{n}\, d\Sigma, \tag{16}$$

where $\boldsymbol{n}(\boldsymbol{\xi})$ is the unit vector of the outward normal to Σ:

$$\boldsymbol{n}(\boldsymbol{\xi}(\vec{q})) = \partial_{q_1}\boldsymbol{\xi}(\vec{q}) \times \partial_{q_2}\boldsymbol{\xi}(\vec{q})/\sqrt{g_0(\vec{q})}, \quad \sqrt{g_0(\vec{q})} = |\partial_{q_1}\boldsymbol{\xi}(\vec{q}) \times \partial_{q_2}\boldsymbol{\xi}(\vec{q})| \quad \text{for} \quad \vec{q} \in Q,$$

$\boldsymbol{\xi} = \boldsymbol{\xi}(\vec{q})$ is local parametrization of Σ.

Using Eq. (16), we write the first integral in the left side of equality (15) in parametric form as

$$\int\limits_{\Sigma} p(\boldsymbol{\xi})\boldsymbol{k}(\boldsymbol{\xi}) \cdot \boldsymbol{n}(\boldsymbol{\xi})\, d\Sigma = \int\limits_{Q} \tilde{p}(\vec{q})\tilde{\boldsymbol{k}}(\vec{q}) \cdot \tilde{\boldsymbol{n}}(\vec{q})\, \sqrt{g_0(\vec{q})}\, d\vec{q},$$

where $\tilde{p}(\vec{q}) = p(\boldsymbol{\xi}(\vec{q}))$, $\tilde{\boldsymbol{k}}(\vec{q}) = \boldsymbol{k}(\boldsymbol{\xi}(\vec{q}))$, and $\tilde{\boldsymbol{n}}(\vec{q}) = \boldsymbol{n}(\boldsymbol{\xi}(\vec{q}))$. Then, Eq. (15) acquires the form

$$\int\limits_{Q} \tilde{p}(\vec{q})\tilde{\boldsymbol{k}}(\vec{q}) \cdot \tilde{\boldsymbol{n}}(\vec{q})\, \sqrt{g_0(\vec{q})}\, d\vec{q} + \int\limits_{\Sigma} (M \circ \Phi)\, \boldsymbol{k} \circ \Phi \cdot \boldsymbol{\nu}(\boldsymbol{x} \circ \Phi)\lambda(\boldsymbol{\xi})\, d\Sigma = 0. \tag{17}$$

To simplify the integrand of the second term in the left side of this equality, we find the relation between the vectors $\boldsymbol{n}(\boldsymbol{\xi})$ and $\boldsymbol{\nu}(\boldsymbol{x}(\boldsymbol{\xi}))$. Let $\boldsymbol{X} = \boldsymbol{X}(\vec{q}) = \boldsymbol{x}(\boldsymbol{\xi}(\vec{q}))$. We recall that

$$\boldsymbol{\nu}(\boldsymbol{x}(\boldsymbol{\xi}(\vec{q}))) = \partial_{q_1}\boldsymbol{X}(\vec{q}) \times \partial_{q_2}\boldsymbol{X}(\vec{q})/\sqrt{g^{\boldsymbol{x}}(\vec{q})}, \qquad \sqrt{g^{\boldsymbol{x}}(\vec{q})} = |\partial_{q_1}\boldsymbol{X}(\vec{q}) \times \partial_{q_2}\boldsymbol{X}(\vec{q})|.$$

Then,

$$\partial_{q_i}\boldsymbol{X}(\vec{q}) = \sum_{k=1}^{3} \partial_{q_i}\xi_k(\vec{q})\, \partial_{\xi_k}\boldsymbol{x}(\boldsymbol{\xi}(\vec{q})),$$

which implies that

$$\partial_{q_1}\boldsymbol{X} \times \partial_{q_2}\boldsymbol{X} = \begin{vmatrix} \partial_{q_1}\xi_1 & \partial_{q_1}\xi_2 \\ \partial_{q_2}\xi_1 & \partial_{q_2}\xi_2 \end{vmatrix} \partial_{\xi_1}\boldsymbol{x} \times \partial_{\xi_2}\boldsymbol{x} + \begin{vmatrix} \partial_{q_1}\xi_3 & \partial_{q_1}\xi_1 \\ \partial_{q_2}\xi_3 & \partial_{q_2}\xi_1 \end{vmatrix} \partial_{\xi_3}\boldsymbol{x} \times \partial_{\xi_1}\boldsymbol{x}$$

$$+ \begin{vmatrix} \partial_{q_1}\xi_2 & \partial_{q_1}\xi_3 \\ \partial_{q_2}\xi_2 & \partial_{q_2}\xi_3 \end{vmatrix} \partial_{\xi_2}\boldsymbol{x} \times \partial_{\xi_3}\boldsymbol{x} = \Big[\partial_{\xi_2}\boldsymbol{x} \times \partial_{\xi_3}\boldsymbol{x};\, \partial_{\xi_3}\boldsymbol{x} \times \partial_{\xi_1}\boldsymbol{x};\, \partial_{\xi_1}\boldsymbol{x} \times \partial_{\xi_2}\boldsymbol{x}\Big](\partial_{q_1}\boldsymbol{\xi}(\vec{q}) \times \partial_{q_2}\boldsymbol{\xi}(\vec{q})).$$

Note that

$$(M^*)^{-1} = \Big[\partial_{\xi_2}\boldsymbol{x} \times \partial_{\xi_3}\boldsymbol{x};\, \partial_{\xi_3}\boldsymbol{x} \times \partial_{\xi_1}\boldsymbol{x};\, \partial_{\xi_1}\boldsymbol{x} \times \partial_{\xi_2}\boldsymbol{x}\Big].$$

Hence, we obtain

$$M^*(\partial_{q_1}\boldsymbol{X}(\vec{q}) \times \partial_{q_2}\boldsymbol{X}(\vec{q})) = (\partial_{q_1}\boldsymbol{\xi}(\vec{q}) \times \partial_{q_2}\boldsymbol{\xi}(\vec{q})),$$

which yields the relation between the vectors $\boldsymbol{n}(\boldsymbol{\xi})$ and $\boldsymbol{\nu}(\boldsymbol{x}(\boldsymbol{\xi}))$:

$$M^*(\boldsymbol{\xi}(\vec{q}))\,\boldsymbol{\nu}(\boldsymbol{x}(\boldsymbol{\xi}(\vec{q}))) = \boldsymbol{n}(\boldsymbol{\xi}(\vec{q}))\sqrt{\frac{g_0(\vec{q})}{g^{\boldsymbol{x}}(\vec{q})}}.$$

In this equality, we replace parametrization $\vec{q} \to \Psi(\vec{q})$, where the diffeomorphism $\Psi\colon\ Q \to Q$ satisfies the identity $\Phi(\boldsymbol{\xi}(\vec{q})) = \boldsymbol{\xi}(\Psi(\vec{q}))$. As a result, we obtain the relation

$$(M^* \circ \Phi(\boldsymbol{\xi}(\vec{q})))\,\boldsymbol{\nu}(\boldsymbol{x} \circ \Phi(\boldsymbol{\xi}(\vec{q}))) = \boldsymbol{n}(\boldsymbol{\xi}(\Psi(\vec{q})))\sqrt{\frac{g_0(\Psi(\vec{q}))}{g^{\boldsymbol{x}}(\Psi(\vec{q}))}}.$$

Multiplying both sides of this relation in a scalar manner by $\boldsymbol{k} \circ \Phi$, we obtain

$$(M \circ \Phi)\,\boldsymbol{k} \circ \Phi\ \cdot\ \boldsymbol{\nu}(\boldsymbol{x} \circ \Phi) = \boldsymbol{k}(\boldsymbol{\xi}(\Psi(\vec{q})))\ \cdot\ \boldsymbol{n}(\boldsymbol{\xi}(\Psi(\vec{q})))\sqrt{\frac{g_0(\Psi(\vec{q}))}{g^{\boldsymbol{x}}(\Psi(\vec{q}))}}.$$

The second integral in the left side of Eq. (17) can be presented as an integral with respect to the parameter $\vec{q}$:

$$\int\limits_{\Sigma}((M \circ \Phi)\,\boldsymbol{k} \circ \Phi)\ \cdot\ \boldsymbol{\nu}(\boldsymbol{x} \circ \Phi)\lambda(\boldsymbol{\xi})\,d\Sigma$$

$$= \int\limits_{Q} \boldsymbol{k}(\boldsymbol{\xi}(\Psi(\vec{q})))\ \cdot\ \boldsymbol{n}(\boldsymbol{\xi}(\Psi(\vec{q})))\sqrt{\frac{g_0(\Psi(\vec{q}))}{g^{\boldsymbol{x}}(\Psi(\vec{q}))}}\sqrt{g_0(\vec{q})}\,\lambda(\boldsymbol{\xi}(\vec{q}))\,d\vec{q}. \tag{18}$$

We apply the replacement of the variables $\vec{q} \to \vec{r} = \Psi(\vec{q})$ in the integrand in the right side of this equality. In the new variables, the function $\boldsymbol{\xi}(\vec{q})$ takes the form $\Xi(\vec{r}) = \boldsymbol{\xi} \circ \Psi^{-1}(\vec{r})$. As $d\Sigma$ is a geometric invariant, we obtain

$$d\Sigma = \sqrt{g_0(\vec{q})}\,d\vec{q} = \sqrt{G_0(\vec{r})}\,d\vec{r}, \quad \sqrt{G_0(\vec{r})} = |\partial_{r_1}\Xi(\vec{r}) \times \partial_{r_2}\Xi(\vec{r})|.$$

From here and from Eq. (18), we find that

$$\int\limits_{\Sigma}(M \circ \Phi)\,\boldsymbol{k} \circ \Phi\ \cdot\ \boldsymbol{\nu}(\boldsymbol{x} \circ \Phi)\,\lambda(\boldsymbol{\xi})\,d\Sigma$$

$$= \int\limits_{Q} \boldsymbol{k}(\boldsymbol{\xi}(\vec{r}))\ \cdot\ \boldsymbol{n}(\boldsymbol{\xi}(\vec{r}))\sqrt{\frac{g_0(\vec{r})}{g^{\boldsymbol{x}}(\vec{r})}}\sqrt{G_0(\vec{r})}\lambda(\boldsymbol{\xi}(\Psi^{-1}(\vec{r})))\,d\vec{r}. \tag{19}$$

Replacing $\vec{r}$ by $\vec{q}$ in Eq. (19), we write Eq. (17) in the following form:

$$\int\limits_{Q} \tilde{\boldsymbol{k}}(\vec{q}) \cdot \tilde{\boldsymbol{n}}(\vec{q})\,\sqrt{g_0(\vec{q})}\left(\tilde{p}(\vec{q}) + \sqrt{\frac{G_0(\vec{q})}{g^{\boldsymbol{x}}(\vec{q})}}\ \lambda(\boldsymbol{\xi}(\Psi^{-1}(\vec{q})))\right)d\vec{q} = 0.$$

As $\boldsymbol{k}$ is an arbitrary solenoidal vector field, we have

$$\int\limits_{\Sigma} \boldsymbol{k}(\boldsymbol{\xi}) \cdot \boldsymbol{n}(\boldsymbol{\xi})\,d\Sigma = \int\limits_{Q} \tilde{\boldsymbol{k}}(\vec{q}) \cdot \tilde{\boldsymbol{n}}(\vec{q})\,\sqrt{g_0(\vec{q})}\,d\vec{q} = 0.$$

Hence, there exists an unknown constant C such that

$$\lambda(\boldsymbol{\xi}(\Psi^{-1}(\vec{q}))) = -\sqrt{\frac{g^{\boldsymbol{x}}(\vec{q})}{G_0(\vec{q})}}\,(p(\boldsymbol{\xi}(\vec{q})) + C).$$

In the latter equality, we apply the replacement of the variables $\vec{q} \to \Psi(\vec{q})$:

$$\lambda(\boldsymbol{\xi}(\vec{q})) = -\sqrt{\frac{g^{\boldsymbol{x}}(\Psi(\vec{q}))}{G_0(\Psi(\vec{q}))}}\,(p(\boldsymbol{\xi}(\Psi(\vec{q}))) + C). \tag{20}$$

To finish the proof of Lemma 2, it suffices to establish a relation between the fundamental forms $g^{\boldsymbol{x}}$ and $g^{\boldsymbol{y}}$. In the notation $\boldsymbol{X}(\vec{q}) = \boldsymbol{x}(\boldsymbol{\xi}(\vec{q}))$, $\boldsymbol{Y}(\vec{q}) = \boldsymbol{y}(\boldsymbol{\xi}(\vec{q}))$, and $\Phi(\boldsymbol{\xi}(\vec{q})) = \boldsymbol{\xi}(\Psi(\vec{q}))$, the condition $\boldsymbol{y}(\boldsymbol{\xi}) = \boldsymbol{x} \circ \Phi(\boldsymbol{\xi})$ is written as

$$\boldsymbol{Y}(\vec{q}) = \boldsymbol{X}(\Psi(\vec{q})). \tag{21}$$

By differentiating condition (21) with respect to q_i as

$$\partial_{q_i} \boldsymbol{Y}(\vec{q}) = \partial_{q_i} \Psi_1(\vec{q})\, \partial_{\Psi_1} \boldsymbol{X}(\Psi(\vec{q})) + \partial_{q_i} \Psi_2(\vec{q}) \partial_{\Psi_2} \boldsymbol{X}(\Psi(\vec{q})),$$

we obtain

$$\sqrt{g^{\boldsymbol{y}}(\vec{q})} = \sqrt{g^{\boldsymbol{x}}(\Psi(\vec{q}))}\, \det D_q \Psi(\vec{q}).$$

In the notation used, $\boldsymbol{\xi}(\vec{q}) = \Xi(\Psi(\vec{q}))$, $\vec{r} = \Psi(\vec{q})$, and

$$\partial_{q_1} \boldsymbol{\xi} \times \partial_{q_2} \boldsymbol{\xi} = \partial_{r_1} \Xi \times \partial_{r_2} \Xi \, \det D_q \Psi(\vec{q}),$$

which implies that

$$\sqrt{g_0(\vec{q})} = \sqrt{G_0(\Psi(\vec{q}))}\ \det D_q \Psi(\vec{q}).$$

Thus, we have

$$\frac{\sqrt{g^{\boldsymbol{y}}(\vec{q})}}{\sqrt{g_0(\vec{q})}} = \frac{\sqrt{g^{\boldsymbol{x}}(\Psi(\vec{q}))}}{\sqrt{G_0(\Psi(\vec{q}))}}. \tag{22}$$

Using Eqs. (20) and (22), we finally obtain

$$\lambda(\boldsymbol{\xi}(\vec{q})) = -\sqrt{\frac{g^{\boldsymbol{y}}(\vec{q})}{g_0(\vec{q})}} \Big(p(\boldsymbol{\xi}(\Psi(\vec{q}))) + C\Big) = -\sqrt{\frac{g^{\boldsymbol{y}}(\vec{q})}{g_0(\vec{q})}} \Big(p \circ \Phi(\boldsymbol{\xi}(\vec{q})) + C\Big),$$

which finalizes the proof of Lemma 2.

3. Variation of the Willmore Functional. In calculating the variation, we use the results of [6]. Let $\boldsymbol{y}(\cdot)\colon \Sigma \mapsto \Sigma^{\boldsymbol{y}}$ by a smooth immersion of the manifold Σ in $\mathbb{R}^3$. Let us recall the notation of parametrization of the surface $\Sigma^{\boldsymbol{y}}$: $\boldsymbol{Y}(\vec{q}) = \boldsymbol{y}(\boldsymbol{\xi}(\vec{q}))$, where $\vec{q} = (q^1, q^2)$. Let us consider a new immersion

$$\boldsymbol{Y}(\vec{q}) = \boldsymbol{Y}(\vec{q}) + \delta \boldsymbol{Y}(\vec{q}).$$

We decompose $\delta \boldsymbol{Y}(\vec{q})$ into the tangential and normal components:

$$\delta \boldsymbol{Y} = \delta_{\|} \boldsymbol{Y} + \delta_{\perp} \boldsymbol{Y} = \theta^{\,i} \boldsymbol{Y}_i + \theta \boldsymbol{\nu}.$$

Then, the tangential variation of the functional $\tilde{W}$ has the form

$$\delta_{\|} \tilde{W}(\Sigma^{\boldsymbol{y}}) = \frac{1}{2} \int\limits_{\Sigma^{\boldsymbol{y}}} \frac{1}{\sqrt{g^{\boldsymbol{y}}}}\, \partial_{q^i} (\sqrt{g^{\boldsymbol{y}}}\, \theta^i H^2)\, d\Sigma^{\boldsymbol{y}} = \frac{1}{2} \int\limits_{\Sigma^{\boldsymbol{y}}} \frac{1}{\sqrt{g^{\boldsymbol{y}}}}\, \partial_{q^i} (\sqrt{g^{\boldsymbol{y}}}\, \theta^i H^2)\, d\Sigma^{\boldsymbol{y}} = \frac{1}{2} \int\limits_{\Sigma^{\boldsymbol{y}}} \operatorname{div} (\vec{\theta} H^2)\, d\Sigma^{\boldsymbol{y}},$$

where $\vec{\theta} = (\theta_1, \theta_2)$. As the surface $\Sigma^{\boldsymbol{y}}$ is closed we have $\delta_{\|} \tilde{W}(\Sigma^{\boldsymbol{y}}) = 0$.

The normal variations of H, $\sqrt{g^{\boldsymbol{y}}}$, and $d\Sigma^{\boldsymbol{y}}$ are calculated by the formulas

$$\delta_{\perp} d\Sigma^{\boldsymbol{y}} = -\theta H \, d\Sigma^{\boldsymbol{y}}, \qquad \delta_{\perp} \sqrt{g^{\boldsymbol{y}}} = -\theta H \sqrt{g^{\boldsymbol{y}}}, \qquad \delta_{\perp} H = \Delta_{g^{\boldsymbol{y}}} \theta + L_{ij} L^{ij} \theta,$$

where $\Delta_{g^{\boldsymbol{y}}}$ is the Laplace–Beltrami operator defined by the expression

$$\Delta_{g^{\boldsymbol{y}}} = \frac{1}{\sqrt{g^{\boldsymbol{y}}}}\, \partial_{q^i} (\sqrt{g^{\boldsymbol{y}}} g^{ij} \partial_{q^j}).$$

The normal variation of the functional $\tilde{W}(\Sigma^{\boldsymbol{y}})$ is determined as

$$\delta_{\perp} \tilde{W}(\Sigma^{\boldsymbol{y}}) = \frac{1}{2} \int\limits_{\Sigma^{\boldsymbol{y}}} \delta_{\perp} (H^2 d\Sigma^{\boldsymbol{y}}) = \int\limits_{\Sigma^{\boldsymbol{y}}} H \delta_{\perp} H \, d\Sigma^{\boldsymbol{y}} + \frac{1}{2} \int\limits_{\Sigma^{\boldsymbol{y}}} H^2 \delta_{\perp} (d\Sigma^{\boldsymbol{y}})$$

$$= \int\limits_{\Sigma^{\boldsymbol{y}}} \Big(H(\Delta_{g^{\boldsymbol{y}}} \theta + L_{ij} L^{ij} \theta) - \frac{1}{2}\, \theta H^3\Big)\, d\Sigma^{\boldsymbol{y}}.$$

We simplify the expression to

$$\int_{\Sigma^{\boldsymbol{y}}} H(\Delta_{g^{\boldsymbol{y}}}\theta + L_{ij}L^{ij}\theta)\, d\Sigma^{\boldsymbol{y}} = \int_{\Sigma^{\boldsymbol{y}}} \theta(\Delta_{g^{\boldsymbol{y}}} H + L_{ij}L^{ij}H)\, d\Sigma^{\boldsymbol{y}} + \int_{\Sigma^{\boldsymbol{y}}} \nabla_{q^i}(H\nabla_{q_i}\theta - \theta\nabla_{q_i}H)\, d\Sigma^{\boldsymbol{y}}.$$

As $\Sigma^{\boldsymbol{y}}$ is a closed surface, we have

$$\int_{\Sigma^{\boldsymbol{y}}} \nabla_{q^i}(H\nabla_{q_i}\theta - \theta\nabla_{q_i}H)\, d\Sigma^{\boldsymbol{y}} = 0.$$

Hence,

$$\delta\tilde{W}(\Sigma^{\boldsymbol{y}}) = \int_{\Sigma^{\boldsymbol{y}}} \Big(\Delta_g H + \Big(\frac{1}{2}H^2 + R\Big)H\Big)\theta\, d\Sigma^{\boldsymbol{y}},$$

where R is the scalar curvature of the surface $\Sigma^{\boldsymbol{y}}$ satisfying the Gauss–Codacci equation

$$L_{ij}L^{ij} = H^2 + R.$$

4. Lagrangian Principle. Governing Equations of the Hydroelastic Structure. Now we can derive equations of motion of the hydroelastic structure in the Lagrangian variables. For simplicity, we consider the case where the stored energy of the shell is defined by the Willmore functional. The governing equations of the hydroelastic structure with constraints can be derived with the use of the Lagrangian variational principle

$$\delta L_f - \tilde{N}(\delta\boldsymbol{x}) + \delta L_e - \tilde{L}(\delta\boldsymbol{y}) = 0 \tag{23}$$

for all smooth functions $(\delta\boldsymbol{x}, \delta\boldsymbol{y})$. Here the linear functionals $\tilde{N}$ and $\tilde{L}$ are defined by the equalities

$$\tilde{N}(\delta\boldsymbol{x}) = \int_{\Omega} \boldsymbol{N}(t,\boldsymbol{\xi})\cdot\delta\boldsymbol{x}(t,\boldsymbol{\xi})\, d\boldsymbol{\xi}, \qquad \tilde{L}(\delta\boldsymbol{y}) = \int_{\Sigma} \boldsymbol{L}(t,\boldsymbol{\xi})\cdot\delta\boldsymbol{y}(t,\boldsymbol{\xi})\, d\Sigma$$

($\boldsymbol{N}$ and $\boldsymbol{L}$ are the constraints). Using Lemma 2 and the expressions δL_f, δL_e, and $\delta\tilde{W}$ in explicit form, we write Eq. (23) as

$$-\int_{\Omega} \Big(\partial_t^2\boldsymbol{x}(t,\boldsymbol{\xi}) + (M^{-1})^*\nabla_\xi p(t,\boldsymbol{\xi})\Big)\cdot\delta\boldsymbol{x}\, d\boldsymbol{\xi}$$

$$-\int_{Q} \Big(\partial_t^2\boldsymbol{y}(t,\boldsymbol{\xi}(\vec{q})) + \Big(\Delta_{g_t^{\boldsymbol{y}}}H + \Big(\frac{1}{2}H^2 + R\Big)H\Big)\sqrt{\frac{g_t^{\boldsymbol{y}}}{g_0}}\,\boldsymbol{\nu}(t,\boldsymbol{y}(t,\boldsymbol{\xi}(\vec{q})))$$

$$-\,(p(t,\Phi(t,\boldsymbol{\xi}(\vec{q}))) + C(t))\sqrt{\frac{g_t^{\boldsymbol{y}}}{g_0}}\,\boldsymbol{\nu}(t,\boldsymbol{y}(t,\boldsymbol{\xi}(\vec{q})))\Big)\cdot\delta\boldsymbol{y}\,\sqrt{g_0}\,d\vec{q} = 0.$$

Within the framework of Hypotheses 1–4 (see Sec. 1), the equation in variations (23) is equivalent to the following boundary-value problem of the dynamics of the hydroelastic structure.

Problem A. We have to find time-dependent diffeomorphisms $\boldsymbol{x}(t,\cdot)\colon\ \Omega \mapsto \Omega_t \subset \mathbb{R}^3$ and $\boldsymbol{y}(t,\cdot)\colon\ \Sigma \mapsto \Sigma_t^{\boldsymbol{y}} \subset \mathbb{R}^3$, a function $p(t,\cdot)\colon\ \Omega \mapsto \mathbb{R}$, a function $C(t)$, and a family of diffeomorphisms $\Phi(t,\cdot)\colon\ \Sigma \mapsto \Sigma$ satisfying the following equations:

$$\boldsymbol{x}(t,\Phi(t,\boldsymbol{\xi})) = \boldsymbol{y}(t,\boldsymbol{\xi}) \qquad \text{for} \quad \boldsymbol{\xi}\in\Sigma; \tag{24a}$$

$$\boldsymbol{\nu}(t,\boldsymbol{x}(t,\Phi(t,\boldsymbol{\xi})))\cdot\partial_t\boldsymbol{x}(t,\Phi(t,\boldsymbol{\xi})) = \boldsymbol{\nu}(t,\boldsymbol{y}(t,\boldsymbol{\xi}))\cdot\partial_t\boldsymbol{y}(t,\boldsymbol{\xi}) \qquad \text{for} \quad \boldsymbol{\xi}\in\Sigma; \tag{24b}$$

$$\partial_t^2\boldsymbol{x}(t,\boldsymbol{\xi}) + (M^{-1}(t,\boldsymbol{\xi}))^*\,\nabla_\xi p(t,\boldsymbol{\xi}) = 0, \qquad \det M = \det D_\xi\boldsymbol{x}(t,\boldsymbol{\xi}) \equiv 1 \qquad \text{for} \quad \boldsymbol{\xi}\in\Omega; \tag{24c}$$

$$\begin{aligned}&\varrho(t,\boldsymbol{\xi})\partial_t^2\boldsymbol{y}(t,\boldsymbol{\xi}) + (\Delta_{g_t^{\boldsymbol{y}}}H + (H^2/2 + R)H)\boldsymbol{\nu}(t,\boldsymbol{y}(t,\boldsymbol{\xi}))\\ &\quad = (p(t,\Phi(t,\boldsymbol{\xi})) + C(t))\boldsymbol{\nu}(t,\boldsymbol{y}(t,\boldsymbol{\xi})) \qquad \text{for} \quad \boldsymbol{\xi}\in\Sigma;\end{aligned} \tag{24d}$$

$$\boldsymbol{x}(0,\boldsymbol{\xi}) = \boldsymbol{\xi} \quad \text{for} \quad \boldsymbol{\xi}\in\Omega, \qquad \boldsymbol{y}(0,\boldsymbol{\xi}) = \boldsymbol{\xi}, \quad \varrho(0,\boldsymbol{\xi}) \equiv 1 \quad \text{for} \quad \boldsymbol{\xi}\in\Sigma.$$

Here $H = H(t, \boldsymbol{y}(t, \boldsymbol{\xi}))$ is the doubled mean curvature $\Sigma_t^{\boldsymbol{y}}$ at the point $\boldsymbol{y}(t, \boldsymbol{\xi})$ and $\boldsymbol{\nu}(t, \boldsymbol{y}(t, \boldsymbol{\xi}))$ is the unit vector of the normal to $\Sigma_t^{\boldsymbol{y}}$ at the point $\boldsymbol{y}(t, \boldsymbol{\xi})$; the density of the elastic membrane is defined by the formula [see Eq. (3)]

$$\varrho(t, \boldsymbol{\xi}) = \frac{d\Sigma}{d\Sigma_t}(t, \boldsymbol{y}(t, \boldsymbol{\xi})) = \sqrt{\frac{g_0(\vec{q})}{g_t^{\boldsymbol{y}}(\vec{q})}}, \tag{24e}$$

and $\vec{q}$ is an arbitrary local parametrization Σ.

5. Formulation of the Problem in the Eulerian Variables. In the Eulerian formulations, the functions depend on the Cartesian coordinates $\boldsymbol{x} = (x_1, x_2, x_3)$ and the time variable t. Let us recall that the fluid occupies the domain Ω_t with the boundary $\Sigma_t^{\boldsymbol{x}}$ at each time instant $t \in [0, T]$. Let

$$Q_T = \bigcup_{t\in(0,T)} \Omega_t \times \{t\}, \qquad S_T = \bigcup_{t\in(0,T)} \Sigma_t^{\boldsymbol{x}} \times \{t\}.$$

The vector of the normal to S_T is denoted by $\boldsymbol{\nu} = \boldsymbol{\nu}(\boldsymbol{x}, t)$, as in the Lagrangian coordinates. We use $\boldsymbol{v}(\boldsymbol{x}, t) = \partial_t \boldsymbol{x}(t, \boldsymbol{\xi}(\boldsymbol{x}, t))$ to denote the fluid velocity and $\boldsymbol{u}(\boldsymbol{y}, t) = \partial_t \boldsymbol{y}(t, \boldsymbol{\xi}(\boldsymbol{y}, t))$ to denote the membrane velocity. Let the material surface S_T of the domain occupied by the fluid be defined by the equation $F(\boldsymbol{x}, t) = 0$. Hence, the fluid velocity satisfies the kinematic condition

$$\partial_t F(\boldsymbol{x}, t) + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} F(\boldsymbol{x}, t) = 0 \qquad \text{for} \quad F(\boldsymbol{x}, t) = 0.$$

In the Eulerian formulation, it makes no sense to distinguish between $\boldsymbol{x}$ and $\boldsymbol{y}$; hence, in what follows, we use $\boldsymbol{u}(\boldsymbol{x}, t)$ instead of $\boldsymbol{u}(\boldsymbol{y}, t)$. Then, Eq. (24b) is written in the form

$$\boldsymbol{v}(\boldsymbol{x}, t) \cdot \boldsymbol{\nu}(\boldsymbol{x}, t) = \boldsymbol{u}(\boldsymbol{x}, t) \cdot \boldsymbol{\nu}(\boldsymbol{x}, t) \qquad \text{for} \quad (\boldsymbol{x}, t) \in S_T.$$

As $\boldsymbol{\nu} \times \nabla_{\boldsymbol{x}} F = \boldsymbol{0}$ and $\boldsymbol{v} \cdot \boldsymbol{\nu} = \boldsymbol{u} \cdot \boldsymbol{\nu}$, we obtain the second kinematic condition

$$\partial_t F(\boldsymbol{x}, t) + \boldsymbol{u} \cdot \nabla_{\boldsymbol{x}} F(\boldsymbol{x}, t) = 0 \qquad \text{for} \quad F(\boldsymbol{x}, t) = 0.$$

In the new notation, system (24c) is written in the form of the classical system of the Euler equations for the dynamics of an ideal fluid

$$\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} \boldsymbol{v} + \nabla_{\boldsymbol{x}} p = 0, \qquad \operatorname{div}_{\boldsymbol{x}} \boldsymbol{v} = 0 \qquad \text{for} \quad (\boldsymbol{x}, t) \in Q_T.$$

Equation (24d) for $\boldsymbol{u}$ acquires the form

$$\rho \partial_t \boldsymbol{u} + \rho\, \boldsymbol{u} \cdot \nabla_{\boldsymbol{x}} \boldsymbol{u} + (\Delta_{g_t^{\boldsymbol{x}}} H + (H^2/2 + R)H)\boldsymbol{\nu} = (p + C(t))\boldsymbol{\nu} \quad \text{for} \quad F(\boldsymbol{x}, t) = 0.$$

Let us derive the transport equation for density. For this purpose, we involve several auxiliary facts of differential geometry. If

$$\boldsymbol{x}(t + \tau, \boldsymbol{\xi}) = \boldsymbol{x}(t, \boldsymbol{\xi}) + \tau \boldsymbol{u}(\boldsymbol{x}(t, \boldsymbol{\xi}), t) + O(\tau^2),$$

then the first variation of the surface area is written as

$$\frac{d}{d\tau} \sqrt{g^{\boldsymbol{x}}(t + \tau, \boldsymbol{\xi})}\,\Big|_{\tau=0} = \operatorname{tr}\{\boldsymbol{S}(\boldsymbol{x}, t) D_{\boldsymbol{x}} \boldsymbol{u}(\boldsymbol{x}, t)^* \boldsymbol{S}(\boldsymbol{x}, t)\} \sqrt{g_t^{\boldsymbol{x}}},$$

where $\boldsymbol{S}(\boldsymbol{x}, t) = \boldsymbol{I} - \boldsymbol{\nu}(\boldsymbol{x}, t) \otimes \boldsymbol{\nu}(\boldsymbol{x}, t)$. Using the notation

$$\operatorname{div}_{\Sigma_t^{\boldsymbol{x}}} \boldsymbol{u} = \operatorname{tr}\{\boldsymbol{S}(\boldsymbol{x}, t) D_{\boldsymbol{x}} \boldsymbol{u}(\boldsymbol{x}, t)^* \boldsymbol{S}(\boldsymbol{x}, t)\}, \tag{25}$$

we obtain

$$\partial_t \rho(\boldsymbol{x}, t) + \boldsymbol{u}(\boldsymbol{x}, t) \cdot \nabla_{\boldsymbol{x}} \rho(\boldsymbol{x}, t) = \partial_t \varrho(t, \boldsymbol{\xi})$$

$$= -\frac{\sqrt{g_0}}{g_t^{\boldsymbol{x}}} \partial_t \sqrt{g^{\boldsymbol{x}}(t, \boldsymbol{\xi})} = -\sqrt{\frac{g_0}{g_t^{\boldsymbol{x}}}} \operatorname{div}_{\Sigma_t^{\boldsymbol{x}}} \boldsymbol{u} = -\rho \operatorname{div}_{\Sigma_t^{\boldsymbol{x}}} \boldsymbol{u},$$

which yields the transport equation for density

$$\partial_t \rho(\boldsymbol{x}, t) + \boldsymbol{u}(\boldsymbol{x}, t) \cdot \nabla_{\boldsymbol{x}} \rho(\boldsymbol{x}, t) + \rho \operatorname{div}_{\Sigma_t^{\boldsymbol{x}}} \boldsymbol{u} = 0.$$

As a result, we obtain the following problem equivalent to Problem A.

Problem B. We have to find a curvilinear cylinder Q_T with the side boundary S_T, vector fields $\boldsymbol{v}$: $Q_T \mapsto \mathbb{R}^3$, $\boldsymbol{u}$: $S_T \mapsto \mathbb{R}^3$, and functions p: $Q_T \mapsto \mathbb{R}$ and ρ: $S_T \mapsto \mathbb{R}$ satisfying the following equations and boundary conditions:

$$\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} \boldsymbol{v} + \nabla_{\boldsymbol{x}} p = 0, \qquad \operatorname{div}_{\boldsymbol{x}} \boldsymbol{v} = 0 \qquad \text{for} \quad (\boldsymbol{x}, t) \in Q_T,$$
$$\rho \partial_t \boldsymbol{u} + \rho \boldsymbol{u} \cdot \nabla_{\boldsymbol{x}} \boldsymbol{u} + \Big(\Delta_{g_t^{\boldsymbol{x}}} H + (H^2/2 + R) H\Big) \boldsymbol{\nu} = (p + C(t)) \boldsymbol{\nu} \qquad \text{for} \quad F(\boldsymbol{x}, t) = 0; \tag{26a}$$

$$\partial_t \rho + \boldsymbol{u} \cdot \nabla_{\boldsymbol{x}} \rho + \rho \operatorname{div}_{\Sigma_t^{\boldsymbol{x}}} \boldsymbol{u} = 0 \qquad \text{for} \quad F(\boldsymbol{x}, t) = 0,$$
$$\partial_t F(\boldsymbol{x}, t) + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} F(\boldsymbol{x}, t) = \partial_t F(\boldsymbol{x}, t) + \boldsymbol{u} \cdot \nabla_{\boldsymbol{x}} F(\boldsymbol{x}, t) = 0 \qquad \text{for} \quad F(\boldsymbol{x}, t) = 0. \tag{26b}$$

Here the equation $F(t, \boldsymbol{x}) = 0$ defines the surface S_T; the operator $\operatorname{div}_{\Sigma_t^{\boldsymbol{x}}}$ is defined by Eq. (25).

These equations have to be supplemented by the initial data:

$$\Omega_t \Big|_{t=0} = \Omega, \qquad \Sigma_t^{\boldsymbol{x}} \Big|_{t=0} = \Sigma,$$

$$\boldsymbol{v}(\boldsymbol{x}, 0) = \boldsymbol{v}_0(\boldsymbol{x}), \qquad \operatorname{div}_{\boldsymbol{x}} \boldsymbol{v}_0 = 0, \qquad \boldsymbol{x} \in \Omega,$$

$$\boldsymbol{S}(\boldsymbol{x}, 0) \boldsymbol{u}(\boldsymbol{x}, 0) = \boldsymbol{u}_s(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Sigma,$$

$$\rho(\boldsymbol{x}, 0) = \rho_0(\boldsymbol{x}), \qquad \boldsymbol{x} \in \Sigma.$$

The presence of the potential mass forces

$$\boldsymbol{f}(\boldsymbol{x}) = -\nabla_{\boldsymbol{x}} \Pi(\boldsymbol{x})$$

does not exert any significant effect on the form of the equations. In this case, the Lagrangians L_f and L_e have the form

$$L_f = \frac{1}{2} \int_{\Omega} |\partial_t \boldsymbol{x}(t, \boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} - \int_{\Omega} \Pi(\boldsymbol{x}(t, \boldsymbol{\xi})) \, d\boldsymbol{\xi},$$

$$L_e = \frac{1}{2} \int_{\Sigma} |\partial_t \boldsymbol{y}(t, \boldsymbol{\xi})|^2 \, d\Sigma - \tilde{W}(\Sigma_t^{\boldsymbol{y}}) - \int_{\Sigma} \Pi(\boldsymbol{y}(t, \boldsymbol{\xi})) \, d\Sigma.$$

Hence, in the Eulerian formulation (26), Eqs. (26a) are replaced by the equations

$$\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} \boldsymbol{v} + \nabla_{\boldsymbol{x}} p + \nabla_{\boldsymbol{x}} \Pi = 0, \qquad \operatorname{div}_{\boldsymbol{x}} \boldsymbol{v} = 0 \qquad \text{for} \quad (\boldsymbol{x}, t) \in Q_T,$$

$$\rho \, \partial_t \boldsymbol{u} + \rho \boldsymbol{u} \cdot \nabla_{\boldsymbol{x}} \boldsymbol{u} + \Big(\Delta_{g_t^{\boldsymbol{x}}} H + (H^2/2 + R) H\Big) \boldsymbol{\nu} + \rho \nabla_{\boldsymbol{x}} \Pi = (p + C(t)) \boldsymbol{\nu} \quad \text{for} \quad (\boldsymbol{x}, t) \in S_T.$$

6. Two-Dimensional Motion. In the case of two spatial variables, the equations of motion become much simpler. With allowance for applications to the problem of surface waves in a pool covered by an elastic film, we assume that the domain Ω_t occupied by the fluid has the form $\Omega_t = \{\vec{x} = x_1 \vec{i} + x_2 \vec{j},\ x_2 < \eta(x_1, t)\}$, where $\eta = \eta(x_1, t)$ is a function periodic with respect to the variable x_1. The surface $\Sigma_t = \{\vec{x},\ x_2 = \eta(x_1, t)\}$ is unknown and is determined in the course of solving the problem.

By virtue of the assumed absence of separation of the shell from the free surface of the fluid [in the plane $(\vec{i}, \vec{j})$], the free surface of the shell admits parametrization

$$\Sigma_t = \{\vec{y}\colon \vec{y} = \vec{r}(t, s),\ s \in \mathbb{R}\},$$

where the displacement vector $\vec{r}(t, s)$ is a periodic function of the Lagrangian variable s.

We consider auxiliary vectors $\vec{a}$ and $\vec{b}$ and new functions α and β:

$$\vec{a} = \frac{\partial_s \vec{r}}{|\partial_s \vec{r}|}, \qquad \vec{b} = \frac{\partial_s \vec{a}}{|\partial_s \vec{a}|}, \qquad \alpha = |\partial_s \vec{r}\,|, \qquad \beta = |\partial_s \vec{a}\,|.$$

Obviously, $\vec{a} \cdot \vec{b} = \vec{a} \cdot \partial_s \vec{a} = \vec{b} \cdot \partial_s \vec{b} = 0$. We can readily conclude that

$$\partial_s \vec{b} = \vec{a} \vec{a} \cdot \partial_s \vec{b} = -\vec{a} \, \partial_s \vec{a} \cdot \vec{b} = -\vec{a} \, \beta.$$

In this notation, we have

$$\sqrt{g_t^{\boldsymbol{y}}(s)} = \alpha(t, s), \qquad \vec{H} = \frac{1}{\alpha} \, \partial_s \Big(\frac{1}{\alpha} \, \partial_s \vec{r}\Big) = \frac{1}{\alpha} \, \partial_s \vec{a} = \frac{\beta}{\alpha} \, \vec{b}, \qquad |\vec{H}| = \frac{\beta}{\alpha}.$$

Let $\Gamma_t = \{\vec{y} = \vec{r}(t,s),\, 0 \le s \le 2\pi\}$. Let us recall that $\vec{r}(t,s)$ is a periodic function of the variable s. Without losing generality, we assume that the following condition is satisfied.

CONDITION 1. At the initial time, the curve Γ_0 is a straight line:

$$\vec{r}(0,s) = s\vec{i}, \qquad 0 \le s \le 2\pi.$$

The density ϱ_0 is already subjected to the condition of a uniform distribution of matter in the shell at the initial time: $\varrho_0 = 1$. Under these conditions, the law of conservation of mass (24e) means that

$$\varrho(t,s) = \frac{1}{|\partial_s \vec{r}(t,s)|} = \frac{1}{\alpha(t,s)}. \tag{27}$$

Let us consider the Lagrangian function L_e for the membrane

$$L_e = K_e - \tilde{W}(\Gamma_t) - E_p,$$

where K_e is the kinetic energy, $\tilde{W}(\Gamma_t)$ is the stored elastic energy, and E_p is the potential of energy due to gravity.

The kinetic energy of the elastic membrane is determined by the equality

$$K_e = \frac{1}{2}\int\limits_{\Gamma_t} \varrho(t,s)|\partial_t \vec{r}|^2\, d\Gamma_t = \frac{1}{2}\int\limits_{\Gamma_t} \varrho(t,s)|\partial_t \vec{r}|^2 \alpha(t,s)\, ds = \frac{1}{2}\int\limits_0^{2\pi} |\partial_t \vec{r}|^2\, ds.$$

In turn, the stored elastic energy $\tilde{W}(\Gamma_t)$ is presented as

$$\tilde{W}(\Gamma_t) = \frac{1}{2}\int\limits_{\Gamma_t} W\Big(\sqrt{g_t^{\boldsymbol{y}}(s)}, |\vec{H}(t,s)|\Big)\, d\Gamma_t = \frac{1}{2}\int\limits_0^{2\pi} W\Big(\alpha, \frac{\beta}{\alpha}\Big)\alpha\, ds.$$

The function $E(\lambda,\sigma) = W(\lambda,\sigma)\,\lambda$ is subjected to the condition of convexity with respect to the variable λ.

For the gravity field $\vec{g} = -g\vec{j}$ acting in the plane $(\vec{i},\vec{j})$ $[\vec{g} = -\nabla_y \Pi(\vec{y})$ and $\Pi(\vec{y}) = gy_2]$, we calculate the potential of energy due to gravity

$$E_p = \int\limits_{\Gamma_t} \varrho(t,s)\Pi(\vec{r}(t,s))\, d\Gamma_t = \int\limits_0^{2\pi} \varrho(t,s)\Pi(\vec{r}(t,s))\alpha(t,s)\, ds = \int\limits_0^{2\pi} \Pi(\vec{r}(t,s))\, ds.$$

We can easily see that the variations of the functionals K_e and E_p are determined by the equalities

$$\delta E_p = g\int\limits_0^{2\pi} \vec{j}\cdot\delta\vec{r}\, ds, \qquad \delta K_e = -\int\limits_0^{2\pi} \partial_t^2 \vec{r}\cdot\delta\vec{r}\, ds.$$

Calculating the variation of the functional of the stored elastic energy is a more difficult problem and requires special consideration. For brevity, the sign of the dependence on t is omitted.

We express α and β via new unknowns k and q:

$$\alpha = \sqrt{q}, \qquad \beta = \sqrt{qk}.$$

Note that $q = |\partial_s \vec{r}|^2$ and $k = |\partial_s \vec{a}|^2/|\partial_s \vec{r}|^2$. The expression for the integral functional of elastic energy takes the form

$$\tilde{W}(\Gamma_t) = \frac{1}{2}\int\limits_0^{2\pi} F(q,k)\, ds,$$

where $F(q,k) = W\Big(\sqrt{q}, \sqrt{k}\Big)\sqrt{q}$. Then, to find the variation

$$\delta\tilde{W}(\Gamma_t) = \frac{1}{2}\,\delta\int\limits_0^{2\pi} F(q,k)\, ds = \frac{1}{2}\int\limits_0^{2\pi} \Big(\partial_q F\,\delta q + \partial_k F\,\delta k\Big)\, ds$$

it is sufficient to calculate δq and δk by the formulas

$$\delta q = \delta|\partial_s\vec{r}\,|^2 = 2\partial_s\vec{r}\cdot\partial_s\delta\vec{r} = 2\sqrt{q}\,\vec{a}\cdot\partial_s\delta\vec{r},$$

$$\delta k = \delta\,\frac{|\partial_s\vec{a}\,|^2}{|\partial_s\vec{r}\,|^2} = -2\,\frac{|\partial_s\vec{a}\,|^2}{|\partial_s\vec{r}\,|^4}\,\partial_s\vec{r}\cdot\partial_s\delta\vec{r} + 2\,\frac{1}{|\partial_s\vec{r}\,|^2}\,\partial_s\vec{a}\cdot\partial_s\delta\vec{a} = -\frac{2k}{q}\,\partial_s\vec{r}\cdot\partial_s\delta\vec{r}$$

$$+\,\frac{2}{q}\,\partial_s\vec{a}\cdot\partial_s\delta\vec{a} = -\frac{2k}{\sqrt{q}}\,\vec{a}\cdot\partial_s\delta\vec{r} + 2\,\frac{\sqrt{k}}{\sqrt{q}}\,\vec{b}\cdot\partial_s\Big(\frac{1}{\sqrt{q}}\,\partial_s\delta\vec{r} - \frac{\vec{a}}{\sqrt{q}}\,(\vec{a},\partial_s\delta\vec{r}\,)\Big)$$

$$= -\frac{2k}{\sqrt{q}}\,\vec{a}\cdot\partial_s\delta\vec{r} + 2\,\frac{\sqrt{k}}{\sqrt{q}}\,\vec{b}\cdot\partial_s\Big(\frac{1}{\sqrt{q}}\,\partial_s\delta\vec{r}\Big) - 2\,\frac{\sqrt{k}}{\sqrt{q}}\,\vec{b}\cdot\partial_s\Big(\frac{\vec{a}}{\sqrt{q}}\,(\vec{a},\partial_s\delta\vec{r}\,)\Big)$$

$$= -\frac{2k}{\sqrt{q}}\,\vec{a}\cdot\partial_s\delta\vec{r} + 2\,\frac{\sqrt{k}}{\sqrt{q}}\,\vec{b}\cdot\partial_s\Big(\frac{1}{\sqrt{q}}\,\partial_s\delta\vec{r}\Big) - \frac{2k}{\sqrt{q}}\,\vec{a}\cdot\partial_s\delta\vec{r}$$

$$= -\frac{4k}{\sqrt{q}}\,\vec{a}\cdot\partial_s\delta\vec{r} + 2\,\frac{\sqrt{k}}{\sqrt{q}}\,\vec{b}\cdot\partial_s\Big(\frac{1}{\sqrt{q}}\,\partial_s\delta\vec{r}\Big),$$

where

$$\delta\vec{a} = \frac{1}{\sqrt{q}}\,\partial_s\delta\vec{r} - \frac{1}{\sqrt{q}}\,\vec{a}\,\vec{a}\cdot\partial_s\delta\vec{r}, \qquad \partial_s\vec{a} = \sqrt{qk}\,\vec{b}, \qquad \vec{a} = \frac{\partial_s\vec{r}}{\sqrt{q}}.$$

It follows that

$$\frac{1}{2}\,\delta\int_0^{2\pi} F(k,q)\,ds = \int_0^{2\pi}\Big(\Big(\sqrt{q}\,\partial_q F - \frac{2k}{\sqrt{q}}\,\partial_k F\Big)\,\vec{a} - \frac{1}{\sqrt{q}}\,\partial_s\Big(\frac{\sqrt{k}}{\sqrt{q}}\,\partial_k F\,\vec{b}\Big)\Big)\cdot\partial_s\delta\vec{r}\,ds.$$

As $\partial_s\vec{b} = -\sqrt{qk}\,\vec{a}$, this equality can be written in the form

$$\frac{1}{2}\,\delta\int_0^{2\pi} F(k,q)\,ds = \int_0^{2\pi}\Big(\Big(\sqrt{q}\,\partial_q F - \frac{k}{\sqrt{q}}\,\partial_k F\Big)\,\vec{a} - \frac{1}{\sqrt{q}}\,\partial_s\Big(\frac{\sqrt{k}}{\sqrt{q}}\,\partial_k F\Big)\,\vec{b}\Big)\cdot\partial_s\delta\vec{r}\,ds.$$

Let us return to the variables α and β in the right side of this equality. If we introduce a new function $E = E(\alpha,\beta)$ as

$$E(\alpha,\beta) = F(\alpha^2, \beta^2/\alpha^2) \equiv \alpha W(\alpha, \beta/\alpha),$$

then the expressions at the vectors $\vec{a}$ and $\vec{b}$ are turned to

$$\sqrt{q}\,\partial_q F - \frac{k}{\sqrt{q}}\,\partial_k F = \alpha\,\partial_q F - \frac{\beta^2}{\alpha^3}\,\partial_k F = \frac{1}{2}\,\partial_\alpha E(\alpha,\beta), \qquad \frac{\beta}{\alpha^2}\,\partial_q F = \frac{1}{2}\,\partial_\beta E(\alpha,\beta).$$

Finally, we obtain

$$\delta\tilde{W}(\Gamma_t) = -\int_0^{2\pi}\partial_s\Big(U\vec{a} - \frac{1}{\alpha}\,(\partial_s V)\vec{b}\Big)\cdot\delta\vec{r}\,ds,$$

where $U = \partial_\alpha E(\alpha,\beta)$ and $V = \partial_\beta E(\alpha,\beta)$.

Let the domain Ω occupied by the fluid at the initial time have the form $\{(\xi_1,\xi_2)\in\mathbb{R}^2\colon\ \xi_2\leqslant 0\}$. Then, the boundary of the domain Ω is $\mathbb{R}$. Such a choice of the domain Ω is caused by Condition 1. We assume that $\vec{\xi}$-parametrization of the domain boundary $\vec{x}(t,\Omega)$ is chosen so that $|\partial_{\xi_1}\vec{x}(t,\xi_1,0)| = 1$ for $\xi_1\in\mathbb{R}$. Then, the condition of the absence of separation of the elastic shell from the fluid surface takes the form

$$\vec{x}(t,S(t,s),0) = \vec{r}(t,s), \qquad s\in\mathbb{R},$$

where $S(t,\cdot)\colon\mathbb{R}\mapsto\mathbb{R}$ is a family of diffeomorphisms. Let us differentiate this equation with respect to s:

$$\partial_s S(t,s) = \frac{|\partial_s\vec{r}(t,s)|}{|\partial_{\xi_1}\vec{x}(t,S(t,s),0)|} = |\partial_s\vec{r}(t,s)| = \frac{1}{\varrho(t,s)}.$$

In accordance with the Lagrangian variational principle, we obtain

$$\partial_t^2 \vec{r} - \partial_s\Big(U\vec{a} - \frac{(\partial_s V)}{\alpha}\,\vec{b}\Big) + g\vec{j} = \alpha(p(t, S(t, s), 0) + C(t))\vec{b},$$

where $S(t,\cdot)\colon \mathbb{R} \mapsto \mathbb{R}$ is the diffeomorphism. For $p = 0$ and $C = 0$, this equation coincides with Antman's equation [7]. Taking into account Eq. (27) for the density ϱ, we write the last equation in the form

$$\varrho\,\partial_t^2\,\vec{r} - \varrho\,\partial_s(U\vec{a} - \varrho\,(\partial_s V)\,\vec{b}\,) + \varrho\,g\,\vec{j} = (p(t, S(t, s), 0) + C(t))\vec{b}.$$

Repeating the reasoning of Sec. 5, we present the Lagrangian function for the fluid in explicit form:

$$L_f = \frac{1}{2}\int\limits_{\Omega} |\partial_t \vec{x}(t, \vec{\xi}\,)|\, d\vec{\xi} - g\int\limits_{\Omega} x_2(t, \vec{\xi}\,)\, d\vec{\xi}.$$

Following the Lagrangian principle, we obtain

$$\partial_t^2 \vec{x}(t, \vec{\xi}\,) + (M^{-1}(t, \vec{\xi}\,))^* \nabla_\xi p(t, \vec{\xi}\,) + g\vec{j} = 0, \qquad M = D_\xi \vec{x}(t, \vec{\xi}\,).$$

Summarizing the results of this section, we obtain the following problem.

Problem C. We have to find a field of displacements of the fluid $\vec{x}(t, \vec{\xi}\,)$ for $\vec{\xi} \in \Omega$ and the field of displacements of the membrane $\vec{r}(t, s)$, functions $p(t, \vec{\xi}\,)$ and $C(t)$, and diffeomorphism $S(t, \cdot)\colon \mathbb{R} \mapsto \mathbb{R}$ to satisfy the equations

$$\begin{gathered}
\vec{x}(t, S(t, s), 0) = \vec{r}(t, s) \qquad \text{for} \quad s \in \mathbb{R}, \\
\vec{b}(t, s) \cdot \partial_t \vec{x}(t, \xi_1, 0)\Big|_{\xi_1 = S(t,s)} = \vec{b}(t, s) \cdot \partial_t \vec{r}(t, s) \qquad \text{for} \quad s \in \mathbb{R}, \\
\partial_t^2 \vec{x}(t, \vec{\xi}\,) + (M^{-1}(t, \vec{\xi}\,))^* \nabla_\xi p(t, \vec{\xi}\,) + g\vec{j} = 0, \qquad \det M(t, \vec{\xi}\,) = 1 \qquad \text{for} \quad \vec{\xi} \in \Omega, \\
\varrho\,\partial_t^2\,\vec{r} - \varrho\,\partial_s(U\vec{a} - \varrho\,\partial_s V\,\vec{b}\,) + \varrho g\vec{j} = (p(t, S(t, s), 0) + C(t))\,\vec{b} \qquad \text{for} \quad s \in \mathbb{R}, \\
\vec{x}(0, \vec{\xi}\,) = \vec{\xi} \quad \text{for} \quad \xi \in \Omega, \qquad \vec{r}(0, s) = s\,\vec{i}, \quad \varrho(0, s) \equiv 1 \quad \text{for} \quad s \in [0, 2\pi],
\end{gathered} \tag{28}$$

where

$$\begin{gathered}
M = D_\xi \vec{x}(t, \vec{\xi}\,), \qquad \varrho(t, s) = \frac{1}{|\partial_s \vec{r}(t, s)|} = \frac{1}{\partial_s S(t, s)}, \\
U = \partial_\alpha(\alpha W(\alpha, \beta/\alpha)), \qquad V = \partial_\beta(\alpha W(\alpha, \beta/\alpha)), \\
\vec{a} = \frac{\partial_s \vec{r}}{|\partial_s \vec{r}|}, \quad \vec{b} = \frac{\partial_s \vec{a}}{|\partial_s \vec{a}|}, \quad \alpha = |\partial_s \vec{r}|, \quad \beta = |\partial_s \vec{a}|, \quad \Omega = \{(\xi_1, \xi_2) \in \mathbb{R}^2\colon \xi_2 \leqslant 0\}.
\end{gathered}$$

We write Eqs. (28) in the Eulerian coordinates. Note, as the function $\vec{r}(t, s)$ for a fixed t is a 2π-periodic function, we can present the vectors $\vec{a}$ and $\vec{b}$ with the use of a new unknown function, which is the angle of deformation $\theta(t, s)$:

$$\vec{a} = \cos\theta\ \vec{i} + \sin\theta\ \vec{j}, \qquad \vec{b} = -\sin\theta\ \vec{i} + \cos\theta\ \vec{j};$$

thereby, $\partial_s \theta = \beta = |\partial_s \vec{a}|$.

We use S to denote the arc abscissa on the curve Γ_t. Then, we have

$$\Gamma_t = \{\vec{y}\colon\ \vec{y} = \vec{x}(t, S) \equiv \vec{r}(t, s(S, t))\}.$$

As $|\partial_S \vec{x}(t, S)| = 1$, then S is a Eulerian variable. Hence, the expressions for velocity $\vec{u} = \vec{u}(\vec{x}, t)$, density $\rho = \rho(\vec{x}, t)$, and angle of deformation $\tilde{\theta} = \tilde{\theta}(S, t)$ of the elastic membrane have the form

$$\vec{u}(\vec{x}, t)\Big|_{\vec{x} = \vec{x}(t,S)} = \partial_t \vec{r}(t, s)\Big|_{s = s(S,t)},$$

$$\rho(\vec{x}, t)\Big|_{\vec{x} = \vec{x}(t,S)} = \varrho(t, s)\Big|_{s = s(S,t)},$$

$$\tilde\theta(S,t) = \theta(t,s)\Big|_{s=s(S,t)}.$$

We introduce the notation

$$\vec s(S,t) = \vec a(t,s)\Big|_{s=s(S,t)} = \cos(\tilde\theta(S,t))\,\vec i + \sin(\tilde\theta(S,t))\,\vec j,$$

$$\vec n(S,t) = \vec b(t,s)\Big|_{s=s(S,t)} = -\sin(\tilde\theta(S,t))\,\vec i + \cos(\tilde\theta(S,t))\,\vec j.$$

By virtue of the kinematic condition on the free boundary, we have

$$\varrho\,\partial_t^2\,\vec r(t,s)\Big|_{s=\text{const}} = \rho\,(\partial_t\vec u + \vec u\cdot\nabla_x\vec u\,)\Big|_{\vec x=\text{const}}.$$

We can readily see that

$$\alpha(t,s)^{-1}\Big|_{s=s(S,t)} = \rho(\vec x,t)\Big|_{\vec x=\vec x(t,S)} = \partial_S s(S,t), \quad \beta(t,s)\Big|_{s=s(S,t)} = \partial_S\tilde\theta(S,t)\,\rho(\vec x,t)^{-1}\Big|_{\vec x=\vec x(t,S)}.$$

Note that the parametrization $s = s(S,t)$ was chosen with the aim of satisfying the equality

$$\partial_S\Big|_{S=\text{const}} = \frac{1}{\alpha}\,\partial_s\Big|_{s=\text{const}}.$$

In what follows, we use f' instead of $\partial_S f$ to shorten the recording. Hence, we have

$$\frac{1}{\alpha}\,\partial_s\Big(U\vec a - \frac{\partial_s V}{\alpha}\,\vec b\Big) = (P\vec s - Q'\vec n\,)',$$

where $P(\rho,\tilde\theta') = \partial_1 E(1/\rho,\tilde\theta'/\rho)$ and $Q(\rho,\tilde\theta') = \partial_2 E(1/\rho,\tilde\theta'/\rho)$; ∂_1 and ∂_2 denote differentiation with respect to the first and second arguments, respectively.

Similar to derivation of the equations for Problem B, we obtain the following equations. The equation of motion for the membrane in the Eulerian coordinates acquires the form

$$\rho\Big(\partial_t\vec u + \vec u\cdot\nabla_x\,\vec u\,) - (P\vec s - Q'\vec n\,)' + \rho g\vec j = (p(\vec x,t) + C(t))\vec n, \qquad \vec x = \vec x(t,S). \tag{29a}$$

The function ρ satisfies the law of conservation of mass

$$\partial_t\rho + \vec u\cdot\nabla_x\rho + \rho\,\mathrm{div}_{\Gamma_t}\,\vec u = 0 \qquad \text{for} \quad \vec x = \vec x(t,S). \tag{29b}$$

The pressure p and the fluid velocity $\vec v = \vec v(\vec x,t)$ satisfy the equations

$$\partial_t\vec v + \vec v\cdot\nabla_x\vec v + \nabla_x p + g\vec j = 0, \qquad \mathrm{div}_x\,\vec v = 0,$$

where $t \in (0,T)$; $\vec x$ belongs to a curvilinear half-plane bounded by the curve $\vec x = \vec x(t,S)$. Then, the constraint equation (absence of separation) has the form

$$\vec u\cdot\vec n = \vec v\cdot\vec n, \qquad \vec x = \vec x(t,S).$$

7. Steady-State Problem. Let us assume that the sought functions in the Eulerian coordinates are independent of t. As $\vec u\cdot\vec n = \vec v\cdot\vec n = 0$ and $\vec x = \vec x(S)$ on the free boundary, the membrane velocity can be defined as a product of the tangential vector and an unknown scalar function $\tilde u = \tilde u(S)$:

$$\vec u\circ\vec x(S) = \tilde u(S)\vec s(S).$$

Then, the law of conservation of mass (29b) acquires the form

$$(\tilde\rho\tilde u)'(S) = 0, \tag{30a}$$

where $\tilde\rho(S) = \rho\circ\vec x(S)$.

In turn, Eq. (29a) takes the form

$$\tilde\rho\tilde u(\tilde u\vec s\,)' - (P\vec s - Q'\vec n\,)' + \tilde\rho g\vec j = (p(\vec x(S),t) + C(t))\vec n.$$

With allowance that $\vec n' = -\tilde\theta'\vec s$ and $\vec s = \tilde\theta'\vec n$, Eq. (29a) is written in projections as

$$\tilde\rho\tilde u\tilde u' - P' - Q'\tilde\theta' + \tilde\rho g\sin\tilde\theta = 0; \tag{30b}$$

$$\tilde\rho\tilde u^2\tilde\theta' - P\tilde\theta' + Q'' + \tilde\rho g\cos\tilde\theta = p(\vec x(S)) + C. \tag{30c}$$

Equations (30a) and (30b) admit two integrals. The first integral follows from the law of conservation of mass

$$\tilde{\rho}(S)\tilde{u}(S) = C_1 = \text{const.}$$

Taking into account the general form of the functions P and Q,

$$P(\tilde{\rho}, \tilde{\theta}') = \partial_1 E\Big(\frac{1}{\tilde{\rho}}, \frac{\tilde{\theta}'}{\tilde{\rho}}\Big), \qquad Q(\tilde{\rho}, \tilde{\theta}') = \partial_2 E\Big(\frac{1}{\tilde{\rho}}, \frac{\tilde{\theta}'}{\tilde{\rho}}\Big),$$

we can turn the expression $P' + Q'\tilde{\theta}'$ to $\tilde{\rho}\Psi'$, where

$$\Psi(\tilde{\rho}, \tilde{\theta}') = \frac{1}{\tilde{\rho}}\,\partial_1 E\Big(\frac{1}{\tilde{\rho}}, \frac{\tilde{\theta}'}{\tilde{\rho}}\Big) + \frac{\tilde{\theta}'}{\tilde{\rho}}\,\partial_2 E\Big(\frac{1}{\tilde{\rho}}, \frac{\tilde{\theta}'}{\tilde{\rho}}\Big) - E\Big(\frac{1}{\tilde{\rho}}, \frac{\tilde{\theta}'}{\tilde{\rho}}\Big).$$

As $\vec{x}(S) = (x_1(S), x_2(S))$ and $x_2'(S) = \sin\tilde{\theta}(S)$, Eq. (30b) can be written as

$$\tilde{\rho}\,\frac{d}{dS}\Big(\frac{1}{2}\,\tilde{u}^2 + \Psi(\tilde{\rho}, \tilde{\theta}') + gx_2\Big) = 0,$$

whence there follows the second integral

$$\tilde{u}^2/2 = C_2 - \Psi(\tilde{\rho}, \tilde{\theta}') - gx_2.$$

8. Bernoulli Law. The following expressions are valid on the free boundary $\boldsymbol{x} = \boldsymbol{x}(S)$:

$$\begin{gathered} \tilde{\rho}(S)\tilde{u}(S) = C_1, \\ u(S)^2 = C_2 - 2\Psi(\tilde{\rho}(S), \tilde{\theta}'(S)) - 2gx_2(S); \end{gathered} \tag{31a}$$

$$\tilde{\rho}(S)\tilde{u}(S)^2\tilde{\theta}'(S) + (P(\tilde{\rho}(S), \tilde{\theta}'(S))\tilde{\theta}'(S))' - (Q(\tilde{\rho}(S), \tilde{\theta}'(S)))'' + \tilde{\rho}(S)g\cos\tilde{\theta}(S) = p(\boldsymbol{x}(S)) + C \tag{31b}$$

[S is the arc length and $\tilde{\theta}'(S)$ is the curvature]. Resolving the algebraic system (31a) with respect to the quantities $\tilde{\rho}$ and $\tilde{u}$ and substituting them into Eq. (31b), we can obtain one dynamic condition relating the boundary curvature $\tilde{\theta}'$ to the pressure p.

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